

Three Einstein rings: explicit solution and numerical simulation

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ABSTRACT

We investigated the effects of gravitational lensing for a system in which a lens is a point mass and a homogeneous disc with a central hole. In such system there is a variety of cases resulting in formation of one, two and three Einstein rings. We found an explicit solution and considered conditions for existence of the second Einstein ring arising on the disc. Numerical modelling of the images was made for various ratios of the central mass to the disc one and for various values of the disc surface density. We also analysed dependence of the magnification factor on a source position for such system. The result of our work can be used in search of astrophysical objects with a toroidal (ring) structure.

Key words: gravitational lensing; strong.

1 INTRODUCTION

The gravitational lensing phenomenon is presently believed to be an extremely powerful tool to investigate a spatial distribution of matter at different scales in the Universe. The fundamentals of gravitational lensing are well developed and can be found in many books and reviews, such as, e.g., (Bliokh & Minakov 1989; Schneider, Ehlers & Falco 1999; Refsdal & Surdej 1994; Zakharov 1997; Schneider, Kochanek & Wambsganss 2006). Observations have already provided a large gallery of gravitational lenses, and every year is adding several newly discovered objects varying in their features.

The most spectacular effect of gravitational lensing is the Einstein ring, which arises when a point lens happens to be at the line of sight to the source (Einstein 1936). In more complex lens systems, multiple Einstein rings may arise. For example, gravitational lensing by a black hole leads to the formation of multiple Einstein rings (Ohanian 1987; Virbhadrha & Ellis 2000; Virbhadrha 2009; Bisnovatyi-Kogan & Tsupko 2012). Such exotic lens as naked singularity in the scalar field can lead to formation of two Einstein rings (Virbhadrha, Narasimha & Chitre 1998; Virbhadrha & Ellis 2002). In the other case, two Einstein rings can be formed due to lensing by two point masses located at two different distances along the line of sight "source-observer". If a more distant lens is luminous, it may play a role of the second source thus leading to formation of three Einstein rings (Werner et al. 2008). Recently, a lens system with a par-

tial double Einstein ring was discovered – SDSSJ0946+1006 (Gavazzi et al. 2008). The authors suggested that these Einstein rings are formed from two sources situated at different distances from the lens.

Lensing by a ring (without a central mass) was considered in short in the book by Schneider, Ehlers & Falco (1999), where a possibility of formation of the second ring was mentioned, with a notation that the problem of lensing by a ring structure is of "little interest in astrophysics". However, this problem acquires much more importance if a central mass is contained in this lens system. Indeed, the presence of a central mass is a necessary condition for stability of a self-gravitating torus or ring (Bannikova, Vakulik & Sergeev 2012).

The collision of galaxies can lead to the formation of a ring around the central dense core (Gerber, Lamb Balsara, 1996). Such galaxies are known as ring galaxies (Athanasoulas & Bosma, 1985) and typical representative of them is the Hoag's object in which the ring-like distribution of matter around the central galaxy has a perfectly circular shape (Hoag 1950; Schweizer et al. 1987). It is likely that ring structures can be formed in a dark matter. In the galaxy cluster C10024+17, the ring structure has been found in distribution of dark matter with the use of gravitational lensing method (Jee et al. 2007). It was assumed that such ring structure was the result of a collision between two clusters. We may assume that the ring structures of dark matter can also occur in collisions of not only clusters but of galaxies too. Therefore, it is of interest to investigate the effects of gravitational lensing by such systems.

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In this paper, we consider the case where the lens is a central point mass and a ring (hereinafter a disc with a hole) with a homogeneous density distribution. In Section 2 we obtain lens equation and analytical expressions for the Einstein ring radii in the case of a point source. In Section 3 we obtain expressions for the inner and outer boundaries of the Einstein rings (for finite source). In Section 4 we investigate conditions for existence of the second Einstein ring and analyse various cases in which one, two or three Einstein rings are formed. Since the considered case is a generalization of the known cases of lensing by simpler systems, we analyse limiting passage (Section 5). In Section 6, we consider a magnification factor for different parameters of the lensing system. In many cases, we present the images obtained numerically not only for a source located on the line of sight, but also for sources that are offset from it.

2 LENS EQUATION FOR A SYSTEM CONSISTED OF A HOMOGENEOUS DISC AND A CENTRAL MASS

We consider a thin annular disc with inner (R_1) and outer (R_2) radii and a point mass located in the centre of symmetry. The total mass of the system is a sum of the disc mass (M_{disc}) and the central mass (M_0): $M = M_0 + M_{disc}$. Let us denote $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ as coordinate vectors of a source and an image, respectively. The lens equation has the form (Schneider, Ehlers & Falco 1999):

$$\mathbf{y} = \mathbf{x} - \boldsymbol{\alpha}(\mathbf{x}), \quad (1)$$

where $\mathbf{y} = (D_d/D_s)(\boldsymbol{\eta}/\xi_0)$ and $\mathbf{x} = \boldsymbol{\xi}/\xi_0$ are the dimensionless coordinates of the source and image. The radius of the Einstein ring for the total mass is

$$\xi_0 = \sqrt{\frac{4GM}{c^2} \cdot \frac{D_{ds}D_d}{D_s}}, \quad (2)$$

D_s and D_d are distances from observer to the source and lens, respectively; D_{ds} is a distance between the lens and the source. Since the lens is a system consisting of the central mass and the disc, the total deflection angle is

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_{disc}, \quad (3)$$

where the deflection angle due to the central mass is

$$\boldsymbol{\alpha}_0 = m_0 \frac{\mathbf{x}}{x^2}, \quad (4)$$

$m_0 = M_0/M$ is the central mass normalized to the total mass of the system. The angle related to the light deflection due to the gravitational influence of the disc with an arbitrary density distribution has the form

$$\boldsymbol{\alpha}_{disc} = \frac{1}{\pi} \int d^2x' \kappa(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^2}. \quad (5)$$

Here the relative mass of the disc is $m_{disc} = M_{disc}/M$; \mathbf{x}' is the vector up to a square element of the disc and the integration is over total square of the disc. Relative surface density¹ for an arbitrary law of distribution can be represented as $\kappa(\mathbf{x}) = \kappa_0 f(\mathbf{x})$, where $\kappa_0 = m_{disc}/(r_2^2 - r_1^2)$ and

¹ We factored $1/\pi$ out the integral because this factor vanishes after integration over azimuthal angle. So, the number π is absent in the expression for the surface density.

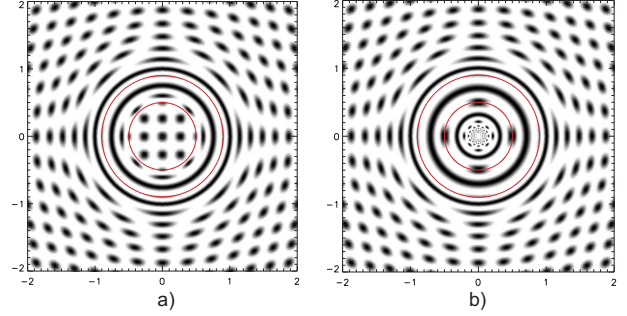


Figure 1. Images of the grid of Gaussian sources (15x15), each of the radius $r_s = 0.1$, formed in gravitational lensing by a) only a homogeneous annular disc $m_{disc} = 1$; b) the central mass $m_0 = 0.1$ and the disc. In both figures the inner radius $r_1 = 0.5$ and the outer radius $r_2 = 0.9$. In all following figures the boundaries of annular lens are shown by the thin red lines.

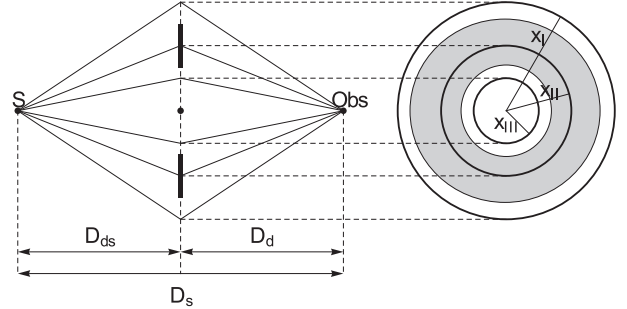


Figure 2. The scheme of formation of the Einstein rings.

$r_{1,2} = R_{1,2}/\xi_0$. In the case of a homogeneous density distribution $\kappa = \text{Const} = \kappa_0$. So, the lens equation (1) for the central mass and the disc has the following form

$$\mathbf{y} = \mathbf{x} - m_0 \frac{\mathbf{x}}{x^2} - \frac{1}{\pi} \int d^2x' \kappa(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^2}, \quad (6)$$

where normalized masses satisfy condition $m_0 + m_{disc} = 1$. Since the considered system is axially symmetric we pass to the polar system of coordinates: $\mathbf{x}' = x' \times (\cos \varphi, \sin \varphi)$ and $\kappa(\mathbf{x}') = \kappa(x')$. After integration over azimuthal angle φ (see Zakharov 1997), the lens equation (6) transforms to three vector equations

$$\mathbf{y} = \mathbf{x} \times \begin{cases} \left(1 - \frac{m_0}{x^2} - \frac{2}{x^2} \int_{r_1}^{r_2} \kappa(x') x' dx' \right), & x \geq r_2 \\ \left(1 - \frac{m_0}{x^2} - \frac{2}{x^2} \int_{r_1}^x \kappa(x') x' dx' \right), & r_1 \leq x \leq r_2 \\ \left(1 - \frac{m_0}{x^2} \right), & x \leq r_1. \end{cases} \quad (7)$$

It is easy to see that in the case of homogeneous density distribution the equation (7) takes the form

$$\mathbf{y} = \mathbf{x} \times \begin{cases} \left(1 - \frac{1}{x^2} \right), & x \geq r_2 & \text{(a)} \\ \left(1 - \frac{m_0}{x^2} - \kappa \frac{x^2 - r_1^2}{x^2} \right), & r_1 \leq x \leq r_2 & \text{(b)} \\ \left(1 - \frac{m_0}{x^2} \right), & x \leq r_1 & \text{(c).} \end{cases} \quad (8)$$

So, we obtained the lens equation (8) for the system consisting of a central mass and a homogeneous annular disc.

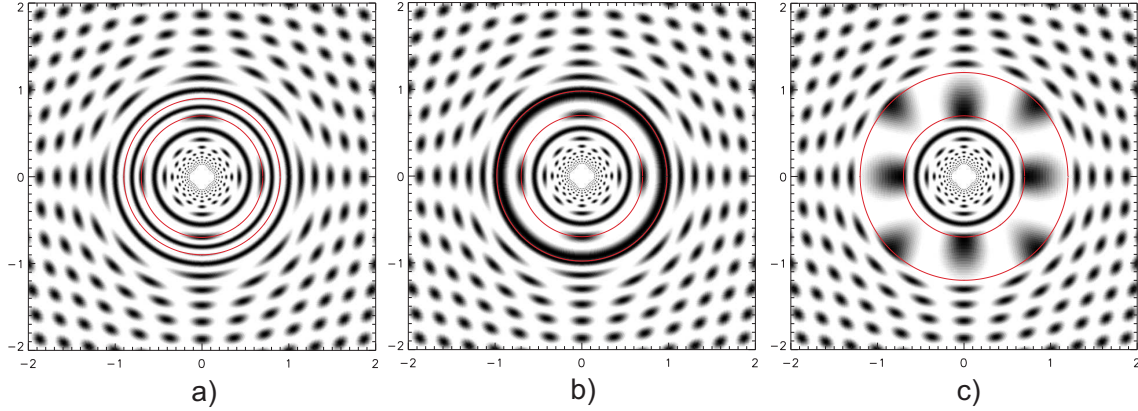


Figure 3. Images of the grid of Gaussian sources ($r_s = 0.1$) due to gravitational lensing by a central point mass with $m_0 = 0.3$ and a homogeneous disc with the inner radius $r_1 = 0.7$, for three cases: a) $r_2 = 0.9$, b) $r_2 = 0.98$, c) $r_2 = 1.2$.

Beyond the outer boundary of disc, the images are identical to those from the point mass equal to the total mass of the system (8a). In the hole of the disc, only the central mass m_0 (8c) influences on the light deflection. And, finally, the disc itself forms images with the characteristics depending mainly on the surface density distribution κ (8b).

We note that in the limiting case $m_0 \rightarrow 0$, $m_{disc} = 1 - m_0 \rightarrow 1$ and the equations (8) transform to the lens equation for the case of a homogeneous disc obtained by Schneider, Ehlers, Falco (1999). Since this case has not been studied in detail, we will consider it briefly in Section 5.

In Fig.1, the images of a grid of sources, appearing due to lensing by a homogeneous annular disc (Fig.1a) and by a system “central mass + disc”, are presented. It is seen from Fig.1a that lensing by the disc leads to formation of two Einstein rings. In this case, the images in the hole of the disc remain undistorted, but the images in the outer region of the disc are distorted as due to lensing by a point mass $m = 1$. The presence of mass in the centre of disc symmetry can lead to formation of three Einstein rings (Fig.1b). Thus, the images have the shape as due to lensing by a point mass m_0 in the region inside the third Einstein ring. (The idea to represent the lensing images for the grid of Gaussian sources was suggested by Victor Vakulik.)

The images in Fig. 1 and all subsequent figures were obtained in the following way. We divide the selected region in the image plane into a grid of coordinates \mathbf{x}_i with a step h . Then, we calculate the corresponding values of coordinates \mathbf{y}_i using the analytical expressions (8) for the lens system with the defined parameters (m_0, r_1, r_2). From the obtained array $(\mathbf{x}_i, \mathbf{y}_i)$, we chose such \mathbf{y}_i for which the difference $|\mathbf{y}_i - \mathbf{r}_i| < h$, where \mathbf{r}_i are the source coordinates. So, we find corresponding coordinates of image \mathbf{x}_i by obtained values \mathbf{y}_i .

2.1 Radii of the Einstein rings (for the case of a point source)

The equations (8) show that with $\mathbf{y} = 0$ (when point source, lense centre and observer are aligned) three Einstein rings can be formed. It is convenient to number the rings in order of decreasing of their radii (Fig.2). Hereafter, the ring of a radius $x_1 = 1$ we call the first (or main) ring (the solution of

the equation (8a)). The first ring forms if $x_1 > r_2$ for $r_2 < 1$. The equation (8b) for $\mathbf{y} = 0$ leads to quadratic equation which is easy to solve relative to x :

$$x_{II} = \sqrt{\left| \frac{\kappa r_1^2 - m_0}{\kappa - 1} \right|} = \sqrt{\left| \frac{\kappa r_2^2 - 1}{\kappa - 1} \right|} \quad (9)$$

Thus, the second Einstein ring is the ring with radius x_{II} which is always formed on a lensing disc ($r_1 < x_{II} < r_2$) and depends on the disc surface density. It is seen from the equation (8c) that in this system the third Einstein ring appears with the radius $x_{III} = \sqrt{m_0}$. The third ring appears in the hole of the disc $x_{III} < r_1$ or $r_1 > \sqrt{m_0}$.

3 WIDTHS OF THE EINSTEIN RINGS AND ITS BOUNDARIES

A finite size of a source has its effect on the resulting image. Overlapping or merging of the rings is possible in this case. Consider this in detail by examining the boundaries of the Einstein rings for each case separately.

a) The inner and outer boundaries of the *main Einstein ring* have the next view:

$$x_1^{Out, In} = \pm \frac{r_s}{2} + \sqrt{\left(\frac{r_s}{2}\right)^2 + 1} \quad (10)$$

where the signs plus/minus correspond to the outer (x_1^{Out}) and inner (x_1^{In}) boundaries of the ring, and r_s is a sources radius. The expression (10) is easy to obtain from equation (8a) with $\mathbf{y} = \pm r_s$. The width of the main Einstein ring equals the source radius: $\Delta x_1 = x_1^{Out} - x_1^{In} = r_s$. It is obvious that the main ring can overlap with the lensing disc because the width of the main ring depends on the radius of the source. The main ring is entirely seen if $r_2 < x_1^{In}$. Otherwise, the main ring can be screened by the disc, either partially ($x_1^{In} < r_2 < x_1^{Out}$) or fully ($r_2 \geq x_1^{Out}$).

In Fig.3, the images of the grid of sources formed by the lensing system “point mass+disc” are presented. Fig.3a corresponds to the case when the main ring does not overlap with the disc ($x_1^{In} \approx 0.95 > r_2$) and all the three Einstein rings are clearly seen. The case of partial overlapping of the main ring with the lensing disc ($x_1^{In} < r_2, x_1^{Out} \approx 1.05 > r_2$) is presented in Fig.3b. In addition, the first and the second

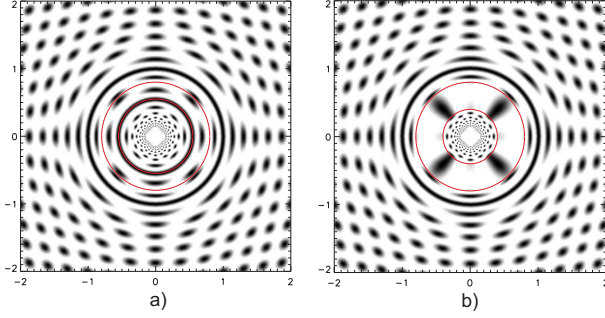


Figure 4. Images of the grid of Gaussian sources ($r_s = 0.1$) due to gravitational lensing by a central point mass with $m_0 = 0.3$ and a homogeneous disc with the inner radius $r_1 = 0.7$, for three cases: a) $r_2 = 0.9$, b) $r_2 = 0.98$, c) $r_2 = 1.2$.

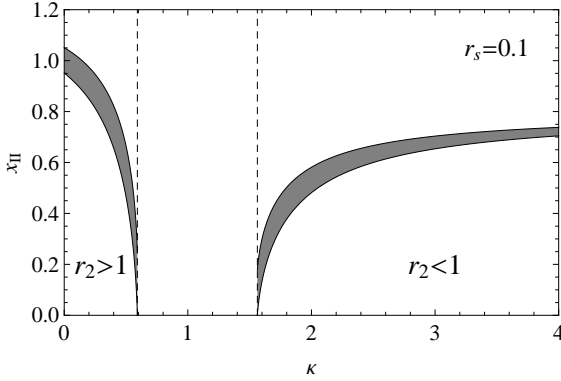


Figure 5. Dependence $x_{II}^{Out,In}(\kappa)$ for two cases: $r_2 = 0.8$ and $r_2 = 1.3$. Grey regions show the change of the second Einstein ring width. “Exclusion region” of the second ring are marked by the dashed lines and corresponds to $\kappa > \kappa_{cr}$ (for $r_2 < 1$) and $\kappa < \kappa_{cr}$ ($r_2 > 1$).

rings merge in one (wide) Einstein ring for the given parameters of the lensing system. As a result, we see two Einstein rings. In Fig.3c, the main ring is fully screened by the disc ($r_2 > x_I^{Out}$). In this case the second Einstein ring vanishes and the resulting image is only one ring (the third Einstein ring). Thus, there is a variety of cases resulting in formation of one, two and three Einstein rings.

b) Generalizing the expression (10) for the case of a point mass m_0 , we can immediately write down the expressions for the inner and outer boundaries of the *third Einstein ring*:

$$x_{III}^{Out,In} = \pm \frac{r_s}{2} + \sqrt{\left(\frac{r_s}{2}\right)^2 + m_0}. \quad (11)$$

We obtain the median radius of the third ring $r_{III} = \sqrt{m_0}$ from (11) for $r_s = 0$. Obviously, the width of the main and the third rings equals the source radius $\Delta x_{III} = \Delta x_I = r_s$. There are also conditions for visibility of the third ring. If the outer radius of the third ring is less than the disc radius $x_{III}^{Out} < r_1$, the third ring is entirely formed (Fig. 3a). A partial overlapping of the third ring by the disc occurs for the following conditions: $x_{III}^{Out} > r_1$, $x_{III}^{In} < r_1$.

The case of partial overlapping of the third Einstein ring is presented in Fig.4a. In this case $x_{III}^{In} = 0.5$ and $x_{III}^{Out} = 0.6$ and the second ring merges with the third one. The param-

eters of the lens system are such that the width of the second ring is almost equal to the width of the third ring. Thus, it is difficult to distinguish the resulting image from that in the case of lensing by two point masses, where two Einstein rings are also formed (Werner, An & Evans 2008). In Fig.4b, the third ring is completely screened by the disc because the condition $x_{III}^{In} > r_1$ is satisfied. Here, we have the case similar to that at Fig.3c, i.e. only one of the three possible rings is formed (the main Einstein ring). The second ring is not formed for the given surface density of the disc. (We will investigate conditions for existence of the second ring in the next section.)

c) As opposed to the main and third rings, boundaries and widths of the *second Einstein ring* are more complicated. The equation (8b) for $y = \pm r_s$ leads to the quadratic equation:

$$(1 - \kappa)x^2 \pm r_s x + \kappa r_2^2 - 1 = 0. \quad (12)$$

Solution of this equation in general case has the form

$$x_{II}^{Out,In} = \frac{1}{2(1 - \kappa)} \left[\pm r_s \pm \sqrt{r_s^2 - 4(1 - \kappa)(\kappa r_2^2 - 1)} \right]. \quad (13)$$

Note that the sign before the square root is determined by the parameters of lensing system. Width of the second ring depends on the lensing disc density and in most cases (see next section) can be written as

$$\Delta x_{II} = \frac{r_s}{|\kappa - 1|}. \quad (14)$$

Thus the width of the second ring depends not only on the source radius but also on the disc surface density (see Figs 3a,b, 4b). Since a source in astrophysical problems is nearly a point one, we will investigate the conditions for existence of the second Einstein ring restricting ourselves to the case $r_s \ll 1$.

4 THE CONDITIONS FOR EXISTENCE OF THE SECOND EINSTEIN RING

It is seen from equation (13) that for $r_s \ll 1$, the expression for the inner and outer boundaries of a ring can be written as

$$x_{II}^{Out,In} = \frac{1}{2|\kappa - 1|} \left[\sqrt{r_s^2 + 4|(\kappa - 1)(\kappa r_2^2 - 1)|} \pm r_s \right] \quad (15)$$

on conditions that a) $\kappa > 1$, $\kappa r_2^2 > 1$ or b) $\kappa < 1$, $\kappa r_2^2 < 1$. These conditions correspond to existence of the real values of the expression (15). It is convenient to analyse the solutions, considering two different cases ($r_2 < 1$ and $r_2 > 1$) and introducing the next notation $\kappa_{cr} = 1/r_2^2$ or $\kappa_{cr} = m_0/r_1^2$. Then the necessary conditions for existence of the second Einstein ring are

$$\begin{aligned} \text{for } r_2 > 1 \text{ it is necessary that } \kappa < \kappa_{cr} \quad (\kappa < 1) \quad a) \\ \text{for } r_2 < 1 \text{ it is necessary that } \kappa > \kappa_{cr} \quad (\kappa > 1) \quad b). \end{aligned} \quad (16)$$

The physical sense of these conditions is a ‘competition’ of gravitational influence between gravitational effects from the central mass and the disc. In Fig.5, dependences of radius and width of the second ring on the surface density for chosen values of the outer radius of the disc $r_2 < 1$ and

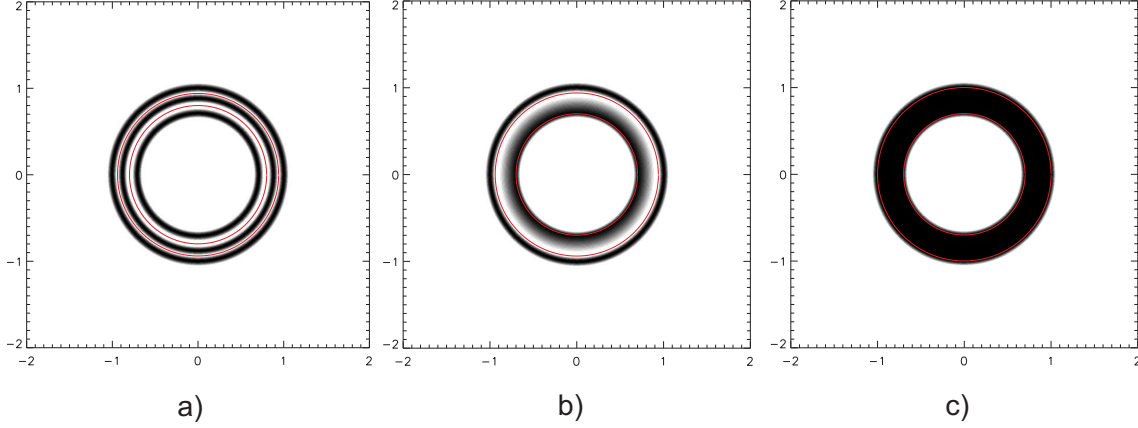


Figure 6. Images of a Gaussian source ($r_s = 0.1$) formed due to gravitational lensing by a central mass $m_0 = 0.5$ and a homogeneous disc with $r_2 < 1$ for the following cases: a) $r_1 = 0.8 > x_{\text{III}}^{\text{Out}}$, $r_2 = 0.94 < x_{\text{I}}^{\text{In}}$, b) $r_1 = 0.7 < x_{\text{III}}^{\text{Out}}$, $r_2 = 0.94$, c) $r_1 = 0.7$, $r_2 = 1.0$.

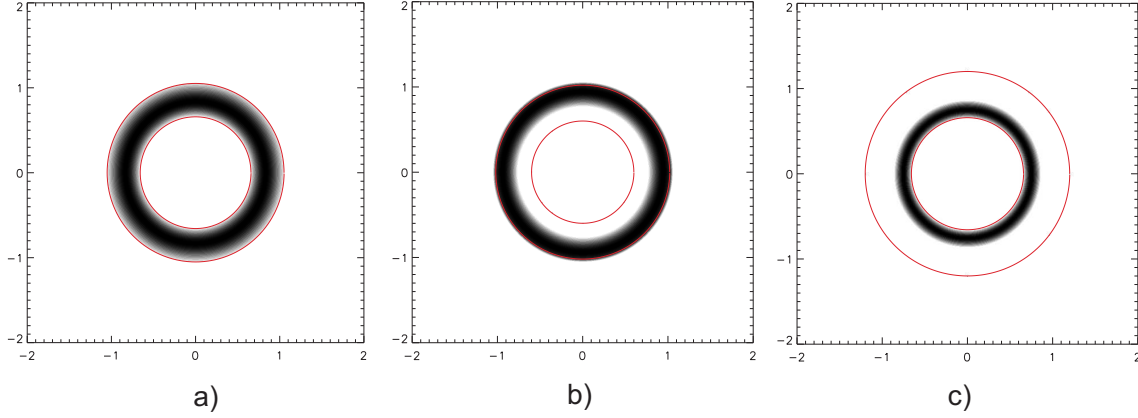


Figure 7. Images of a Gaussian source ($r_s = 0.1$) formed due to gravitational lensing by the central mass $m_0 = 0.5$ and the homogeneous disc with $r_2 < 1$ for the next cases: a) $r_1 = 0.8 > x_{\text{III}}^{\text{Out}}$, $r_2 = 0.94 < x_{\text{I}}^{\text{In}}$, b) $r_1 = 0.7 < x_{\text{III}}^{\text{Out}}$, $r_2 = 0.94$, c) $r_1 = 0.7$, $r_2 = 1.0$.

$r_2 > 1$ are shown. The region of values κ for which the second ring does not exist we call the 'exclusion region'. For $\kappa \gg \kappa_{cr} \gg 1$ ($r_2 < 1$), the width of the second ring is decreased but its radius tends to r_2 asymptotically. This case corresponds to a limiting passage to an infinitely thin ring which we will consider in the next section. On the other hand, for $\kappa \rightarrow 0$ ($r_2 > 1$) the medium radius of the second ring $x_{\text{II}} \rightarrow 1 = x_{\text{I}}$ and width $\Delta x_{\text{II}} \rightarrow r_s$. This case corresponds to the limiting passage to lensing by only the central mass, $m_0 \rightarrow 1$.

The necessary conditions (16) are not sufficient ones, because for real values of the inner and outer radii of the second ring they can go beyond the domain of applicability (8b), i.e. abroad the lensing disc. For example, it is seen from (15) that for $\kappa \rightarrow 1/r_2^2 \pm 0$ the outer radius $x_{\text{II}}^{\text{Out}} \rightarrow \Delta x_{\text{II}}$ and $x_{\text{II}}^{\text{In}} \rightarrow 0$. However if $x_{\text{II}} < r_1$ the second ring is not formed. The sufficient conditions for existence of the second ring are $x_{\text{II}}^{\text{Out}} \leq r_2$ and $x_{\text{II}}^{\text{In}} \geq r_1$. If these two cases are realized simultaneously, the second Einstein ring is seen entirely. In other case, only a part of the second ring is formed due to screening by the disc.

The conditions of the full visibility of the second Einstein ring have the following form: $x_{\text{II}}^{\text{Out}} \leq r_2$ and $x_{\text{II}}^{\text{In}} \geq r_1$. These conditions can be expressed using a minimal set of

parameters (r_s and m_0):

$$\begin{aligned} &\text{for } r_2 \geq 1 \quad r_1 \leq x_{\text{III}}^{\text{In}} \quad \text{and} \quad r_2 \geq x_{\text{I}}^{\text{Out}} \quad \text{a)} \\ &\text{for } r_2 \leq 1 \quad r_1 \geq x_{\text{III}}^{\text{Out}} \quad \text{b)} \end{aligned} \quad (17)$$

If conditions (16b) and (17b) are satisfied (for $r_2 \leq 1$) than the second ring always exist. In this case, the third ring is entirely seen but the first ring can be partly screened by the disc for $r_2 > x_{\text{I}}^{\text{In}}$. Thus, if we define the radius of the source r_s and the value of the central mass m_0 , we can obtain the values r_1 , r_2 , for which three, two or one Einstein ring(s) are formed. In Figs 6 and 7, we presented the results of simulations for $m_0 = 0.5$ and $r_s = 0.1$. In this case, three Einstein rings can exist for the given values of the disc radii (Fig.6a). We note that if the central mass increases (for fixed r_1 , r_2), the third ring is screened by the disc because $x_{\text{III}} = \sqrt{m_0}$. In this case, the radius of the second ring decreases and gradually disappears because $x_{\text{II}}^{\text{Out, In}}$ extends beyond the disc (the condition (17b)). In Fig.6b we can see two Einstein rings. The inner ring is a result of merging the third and partly the second rings. Fig. 6c shows a special case: the surface density in the disc $\kappa = 0.98 \rightarrow 1$. In this case the width of the second ring $\Delta x_{\text{II}} \rightarrow \infty$ (see (14)). As a result, a single Einstein ring is formed with the width equal to the disc width, and the intensity in each point of the ring equals

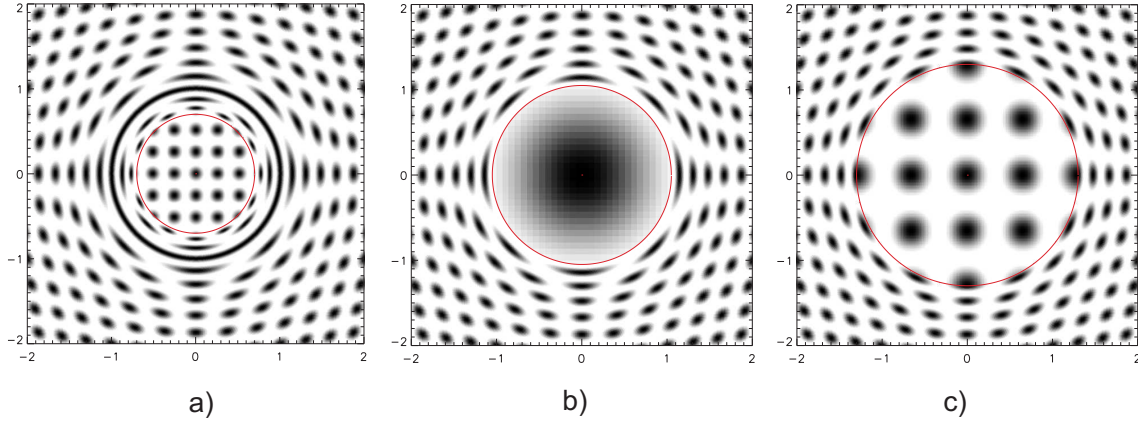


Figure 8. Images of the grid of sources ($r_s = 0.1$) formed due to gravitational lensing by a homogeneous disc ($m_{disc} = 1$ and $r_1 = 0$) for the cases: a) $r = 0.7$, b) $r = 1.05 = x_1^{Out}$, c) $r = 1.3$.

the source intensity at its maximum (the central point of the Gaussian source).

If the outer radius of the disc $r_2 > 1$, then the image is a single Einstein ring, but with different width due to the merging together the first and the second rings or the second and the third rings (Fig.7). Fig.7a shows the limiting case of the conditions (17a) $r_1 = x_{II}^{In}$ and $r_2 = x_1^{Out}$, for which the image is a wide (second) ring entirely filling the disc area entirely. In Fig.7b, the first ring is screened partly by the disc and merges with the second ring. In this case, the conditions (17a) are not satisfied and the outer radius of the second ring x_{II}^{Out} extends beyond the disc, i.e. the second ring is seen partly. In Fig.7c, the second ring contacts the inner boundary of the disc and is visible completely. In this case, the radius of the second ring x_{II} increases up to unity with decreasing m_0 . And vice versa, as the outer radius of the disc (for a given fixed m_0) increases, the radius x_{II} decreases and the second ring is gradually disappearing. Vanishing of the second ring is accompanied by appearance of the third Einstein ring. Such dependence of the second ring radius on the parameters of the lensing system can be understood by considering the limiting cases. We will discuss this in the next section.

Note that in addition to the solution (15) examined above, there is a solution of the form

$$x_{II}^{Out, In} = \frac{1}{2|\kappa - 1|} \left[r_s \pm \sqrt{r_s^2 - 4|(\kappa - 1)(\kappa r_2^2 - 1)|} \right]. \quad (18)$$

This solution is valid for a source with a large radius, so we will not dwell on it. We note only that this solution falls into the range of values for a surface density $1 \leq \kappa \leq \kappa_{cr}$ for $r_2 \leq 1$ and $\kappa_{cr} \leq \kappa \leq 1$ for $r_2 \geq 1$. For a small source these regions are very narrow and abut to the values $\kappa \rightarrow 1$ and/or $\kappa \rightarrow \kappa_{cr}$. For a source with a large value of r_s these regions can work but only in a narrow range of parameters of the lensing system.

We can predict without simulation the configuration of Einstein rings if a central mass in our system is a black hole. In this case, all the features of the Einstein rings, discovered above, will be the same but with additional (relativistic) rings in the nearest vicinity of black hole.

5 LIMITING CASES

The solution, obtained for the lensing system "a homogeneous disc with a hole + central mass", is a generalization of the known cases of more simple systems. Let us consider them briefly because it will be useful for understanding the rings formation in the given system for different lens parameters.

a) Passage to the limit of a point mass $m_0 \rightarrow 1$ ($m_{disc} \rightarrow 0$, $\kappa \rightarrow 0$). It is easy to see that $x_{II} \rightarrow x_I = 1$ and $\Delta x_{II} \rightarrow r_s$, i.e. the second ring degenerates to the main Einstein ring. Thus, if the disc mass is decreasing while keeping its surface area, the radius of the second ring tends to the radius of the main Einstein ring.

b) Passage to the limit of a wide disc $r_2 \gg 1$ also corresponds to $\kappa \rightarrow 0$. Since the central mass has a fixed value $m_0 < 1$, we have $x_{II}^{Out, In} \rightarrow (\sqrt{r_s^2 + 4|\kappa r_1^2 - m_0|} \pm r_s)/2 \rightarrow x_{III}^{Out, In}$ and the second ring degenerates to the third Einstein ring. Note that increasing of the inner radius of the disc results in a decrease of the second ring radius. If the third ring is screened by the disc, we can see only the second ring (Fig.7c). On the other hand, if the value of the central mass decreases, the second ring disappears, because $r_{II} \rightarrow r_{III} = \sqrt{m_0}$. In this case disappearance of the second ring is accompanied by appearance of the third ring.

c) Passage to the limit of an infinitely thin ring $r_1 \rightarrow r_2 \rightarrow r$ corresponds to $\kappa \rightarrow \infty$. It is seen from equations (9) and (14) that in this case the lens equation (8) has the form

$$\mathbf{y} = \mathbf{x} \times \begin{cases} \left(1 - \frac{m_0}{x^2}\right), & x < r \\ \left(1 - \frac{1}{x^2}\right), & x > r \end{cases} \quad (19)$$

If $r < 1$, the first and the third rings exist simultaneously that is possible only in this limiting case. In the system a disc and a central mass, the second ring disappears always in a pair with the first or third ring. For $m_0 = 0$, the solution corresponds to a well-known solution for lensing by an infinitely thin ring (Zakharov 1997). In this case, there is a single Einstein ring and an undistorted image of the source in the centre of the system.

d) Passage to the limit of a solid disc (without hole)

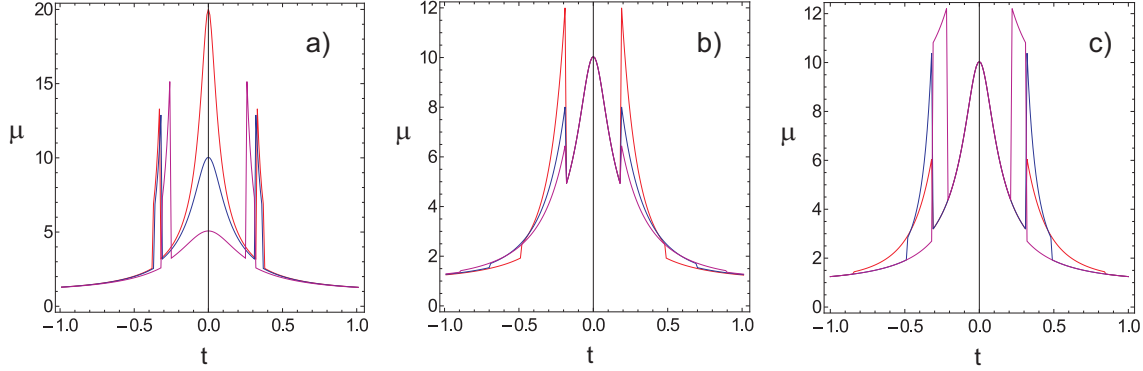


Figure 9. Dependence of magnification factor on a source position a) at the disc with $r_1 = 0.67$, $r_2 = 0.85$ and the central mass with $m_0 = 0.7$ for $r_0 = 0.05$ (orange), 0.1 (blue), 0.2 (magenta); b) at the disc with $r_1 = 0.5$, $r_2 = 0.9$ for the different values of the central mass: $m_0 = 0.5$ (orange); $m_0 = 0.6$ (blue); $m_0 = 0.7$ (magenta); c) $m_0 = 0.5$, $r_2 = 0.85$, $r_1 = 0.4$ (orange), $r_1 = 0.5$ (blue), $r_1 = 0.6$ (magenta).

$r_1 \rightarrow 0$ lead to

$$\mathbf{y} = \mathbf{x} \times \begin{cases} \left(1 - \frac{1}{r^2}\right), & |x| < r \\ \left(1 - \frac{1}{x^2}\right), & |x| > r \end{cases} \quad (20)$$

where we denote $r_2 = r$ for simplicity. This is a well-known solution (Schneider, Ehlers & Falco 1993; Zakharov 1997), but it seems useful to consider the images for some specific cases. It is seen from equation (20a) that the radius of an image formed by the disc increases or decreases depending on the disc radius: $r_{im} = r_s |1 - 1/r^2|^{-1}$. For $r < 1/\sqrt{2}$ the image size is less than the source size (Fig.8a), but for $1/\sqrt{2} < r < 1$ the image of the source increases and tends to infinity for $r \rightarrow 1 \pm 0$. Fig. 8b presents the case where the image is magnified by a factor of about 11 and almost completely fills the disc. Further increasing of the disc radius leads to decreasing of the image size (Fig.8c), and $r_{im} \rightarrow r_s$ for $r \rightarrow \infty$. Similar effect appears in the considered system 'disc + central mass'. Increasing of the image formed by disc (for shifted source) can lead to significant effects on magnification, which we discuss in the next section.

6 MAGNIFICATION FACTOR

Consider a dependence of magnification factor on a source position for the case of a point source. Critical curves are defined by the condition $\det A = 0$ and the magnification factor is $\mu = 1/\det A$, where the matrix determinant $\det A = |\partial y_i / \partial x_j|$. The lens equation (8) gives

$$\det A = \begin{cases} 1 - \frac{1}{x^4}, & x \geq r_2 \\ (1 - \kappa)^2 - \left(\frac{\kappa r_2^2 - 1}{x^2}\right)^2, & r_1 \leq x \leq r_2 \\ 1 - \frac{m_0^2}{x^4}, & x \leq r_1 \end{cases} \quad (21)$$

It is easy to see from (21) that the critical curves are circles with radii equal to the radii of Einstein rings: $x_I^{cr} = x_I = 1$, $x_{II}^{cr} = x_{II} = \sqrt{[(\kappa r_2^2 - 1)/(\kappa - 1)]}$, $x_{III}^{cr} = x_{III} = \sqrt{m_0}$. The caustic (including all three regions) is a point with coordinates $(0, 0)$. The magnification factor for each of the three

regions has the form

$$\begin{aligned} \mu_I^\pm &= \frac{1}{4} \frac{(y \pm \sqrt{y^2 + 4})^2}{y \sqrt{y^2 + 4}} \quad (a) \\ \mu_{II}^\pm &= \frac{1}{4(\kappa - 1)^2} \frac{(y \pm \sqrt{y^2 + 4(\kappa - 1)(\kappa r_2^2 - 1)})^2}{y \sqrt{y^2 + 4(\kappa - 1)(\kappa r_2^2 - 1)}} \quad (b) \\ \mu_{III}^\pm &= \frac{1}{4} \frac{(y \pm \sqrt{y^2 + 4m_0})^2}{y \sqrt{y^2 + 4m_0}} \quad (c) \end{aligned} \quad (22)$$

The sign '+' corresponds to an image of positive parity, the sign '-' corresponds to an image of negative parity. As one would expect, the magnifications for the first (22a) and third (22c) regions present the magnification for the point lens with a mass equal to 1 and m_0 , respectively. Note that the magnification factor (22b) can be obtained by using the following substitution in the expression (22a): in the denominator $4 \rightarrow 4(\kappa - 1)^2$ and under the square root sign $4 \rightarrow 4(\kappa - 1)(\kappa r_2^2 - 1)^2$. The total magnification taking account of all regions is $\mu = \mu_I + \mu_{II} + \mu_{III}$.

Let us consider a source which moves in a straight line at a certain distance from the centre of a gravitational lens. It is convenient to parametrize y in the follow form

$$y = \sqrt{r_0^2 + t^2}$$

where r_0 is the minimal distance the source approaches the origin of a coordinate system, t is the source position relative to the lens centre (for $t = 0$ $y = r_0$). The magnification curves are obtained in the follow way. We calculate values y_\pm for a given t and then calculate the values x_\pm from the lens equation (8) for different regions, and the corresponding magnifications from the formula (22).

In Fig.9, we show various examples of dependence of magnification factor on a source position (name them "the magnification curves" for convenience). Three well-defined peaks are seen in all the three figures. There are differences in the peak shapes from the disc and the central mass. It is seen from Fig.9a that contribution to the magnification due to lensing by the disc is becoming predominant with an increase of r_0 . On the other hand, the highest values of magnification are achieved at the minimum value of the

central mass (Fig.9b). Fig.9c shows that different widths of the disc result in different shapes of the magnification curves.

7 CONCLUSION

Analysis of the Einstein rings arising due to lensing by a disc and a central mass has shown a diversity of possible cases. First, this is formation of three Einstein rings. Assuming that in most of astrophysical objects $m_0 \geq m_{disc}$, we obtain rough limit on the radius of the disc for which three Einstein rings may exist: $r_1 \geq 0.7$ and $r_2 < 1$ at surface density $\kappa > \kappa_{cr}$. The more common case is two Einstein rings which arise in a wide range of parameters of the lensing system. It is shown that in this case the surface brightness of one of the rings is dependent on the surface density of the disc and can decrease to the inner or outer boundary of the disc. Formally, knowing the brightness distribution in a ring, we can restore the density distribution in the disc. The most frequently encountered case is one Einstein ring. We have shown that the resulting image can be indistinguishable from the main Einstein ring or, conversely, it may be of a substantial width, thus filling the disc completely. In fact, the ring distribution of matter in the lens may have the same effect as a point mass. If the annular lens is not detected in the optical light (for example dark matter), one can make wrong conclusions about the mass and its distribution in the lens, as well as about the distance to it. In contrast, the observation of a wide bright Einstein ring can be a criterion for identification of objects with the ring structures. Note also that the source image can be magnified while passing across the disc (in projection on the lens plane). This leads to a significant rise in the magnification curve, which may also be a criterion for detection of such lenses.

For real objects accounting for inhomogeneous density distribution in the annular lens and for possible inclination of the lens plane to the plane of the sky are needed. All these may be the next step in investigations of the gravitational lensing effects by objects having the ring (toroidal) structures.

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